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THEORY OF SYSTEM TYPE FOR LINEAR MULTIVARIABLE SERVOMECHANISMS

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ABSTRACT

A theory of system type for linear multivariable servomechanisms is developed. New characterizations of system type are formulated and two systematic methods for determining system type are derived. Both methods are algorithmic and easier to apply than the techniques given in earlier studies. The first applies to the case where the open-loop system is described by its transfer function matrix, the second where its time domain state-space representation is given. New results are also presented for characterizing and identifying the system type for certain composite systems from a knowledge of the individual subsystem types.

I. INTRODUCTION

In classical servomechanism theory [1, 2], "system type" is a well established and widely used concept for determining the steady-state tracking error characteristics of a closed-loop system from a knowledge of its open-loop transfer function. Specifically, a single-input/single-output unity feedback system (see Fig. 1) whose open-loop transfer function can be put in the form

$$G(s) = \frac{KN(s)}{s^{\ell}D(s)} \tag{1}$$

is said to be a "type ℓ" servomechanism*. In Eq. (1), K is a nonzero real constant; and N(s) and D(s) are monic polynomials in the complex variable s which (i) are of degree m and $n-\ell$, respectively, where $m\leq n$ and $0\leq \ell\leq n$, and (ii) have no common factors or zero roots. If the closed-loop system is stable, it is known [1,2] that the steady-state error

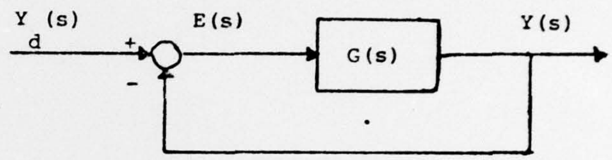
$$\bar{e} = \lim_{t\to\infty} e(t) \triangleq \lim_{t\to\infty} [y_d(t) - y(t)]$$

is zero for all polynomial inputs of the form

$$y_d(t) = \sum_{i=0}^{\ell-1} \alpha_i t^i$$

where $t\geq 0$ and the α_i are arbitrary real constants.

* It is sometimes also convenient [1,2] to refer to G(s) as being type ℓ.



Y_d - input (desired output)
E - error
Y - output
G - transfer function

Fig. 1 Unity feedback system.

The extension of this concept to the case where the system in Fig. 1 is a multivariable (m-input/m-output), m>1, system is a nontrivial matter and has been the subject of a number of investigations [3-8]. The difficulty stems from the interactions of the various inputs with the outputs. This unfortunately precludes identification of system type from the open-loop transfer function by direct inspection as in the scalar case. More fundamentally, there is the issue of what constitutes a useful definition of type for multivariable systems.

Wiberg [3] defines a unity feedback multivariable servomechanism to be type $\ell\geq 1$ if the m-dimensional error vector $e(t) = y_d(t) - y(t)$ has the property that

$$\lim_{t\to\infty} e(t) = 0$$

when the input is

$$y_d(t) = \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix} t^{\ell-1} \tag{2}$$

where $t\geq 0$ and the a_i are arbitrary real constants. He then shows that a necessary and sufficient condition for the closed-loop system to be type ℓ is that the m x m open-loop transfer function matrix G(s) be expressible

$$G(s) = s^{-\ell}R(s) + P(s)$$

where R(s) and P(s) are such that

$$\lim_{s\to 0} sR^{-1}(s) = 0 \quad \text{and} \quad \|\lim_{s\to 0} s^{\ell-1}P(s)\| < \infty$$

where $(\cdot)^{-1}$ and $||\cdot||$ denote matrix inverse and norm, respectively. Except when m is two or three, the above condition is difficult and awkward to apply.

In the approach taken by Sandell and Athans [4] (see also [5]), a unity feedback multivariable servomechanism is said to be type ℓ if ℓ is the largest nonnegative integer for which the open-loop transfer function can be written

$$G(s) = \frac{1}{s^{\ell}} G'(s)$$

where G'(s) has nonsingular d.c. gain. The latter requires that

$$\lim_{s \to 0} \Delta_{G'}(s) \det G'(s) \neq 0$$

where $\Delta_{G'}(s)$ is the characteristic polynomial of G'(s), that is the least common denominator of all minors of G'(s), and det denotes the determinant. This condition is equivalent to G'(s) having no zeros [19] at the origin. Using their definition, Sandell and Athans show that the closed-loop system, if stable, has zero steady-state error for an input given by Eq. (2). Their condition is easier to check than Wiberg's, but also requires a certain amount of trial and error.

Hosoe and Ito [6] were the first to observe that the use of a single number is inadequate for characterizing system type in the multivariable case. For example, if

$$G(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ \frac{1}{s} & \frac{s+2}{s^2} \end{bmatrix}$$

the closed-loop system has zero steady-state error when the input to the system's first "channel" is a step and that to its second channel is a ramp. As a result, an accurate description of system type in this case is $[\ell_1 \;\; \ell_2]=[1 \;\; 2]$ in contrast to both the Wiberg and Sandell-Athans formulations which indicate that the closed-loop system is type 1. The definition of system type thus proposed by Hosoe and Ito is in terms of the vector type $[\ell_1 \ell_2 \ldots \ell_m]$ where $\ell_i \geq 0$ is, for each $i=1,2,\ldots,m$, the system type for the i^{th} input-output pair in the classical sense. Their test for system type consists of first expressing G(s) in a partial fraction expansion

$$G(s) = \sum_{j=1}^{p} \frac{1}{s^{\ell}} G_j + \tilde{G}(s)$$

where G_j are nonzero constant matrices, and $\tilde{G}(s)$ is rational and has no poles at zero. Letting G_j^i be the matrix obtained by deleting the i^{th} row of G_j, one forms

$$N = \begin{bmatrix} G_1 & G_2 & \cdots & G_p \\ G_2 & G_3 & \cdots & 0 \\ \vdots & & & \vdots \\ G_p & 0 & \cdots & 0 \end{bmatrix} \text{ and } N_i = \begin{bmatrix} G_1^i & G_2^i & \cdots & G_p^i \\ G_2^i & G_3^i & \cdots & 0 \\ \vdots & & & \vdots \\ G_p^i & 0 & \cdots & 0 \end{bmatrix}$$

Then, ℓ_i = rank N − rank N_i. This test, which is conceptually simple, can become involved when p is large.

Young and Willems [7] approach the type 1 multivariable servomechanism problem by augmenting the usual time-domain state-space description (A,B,C) by the integral of the error, viz.,

$$z = \int_0^t [y_r(\tau) - y(\tau)]d\tau$$

They then show that the augmented system is controllable if and only if (A,B) is controllable and

$$\text{rank} \begin{bmatrix} A & B \\ -C & 0 \end{bmatrix} = n + m$$

With these conditions satisfied, they are able to assign the closed-loop poles arbitrarily via state variable feedback, and so stabilizing the system guarantee zero steady-state error when y_r is a vector of step inputs. The extent to which any channel of the closed-loop gives zero steady-state error for higher-order inputs, e.g., ramps, parabolas, etc., is unknown. Porter and Bradshaw [8] subsequently extended these results via higher order augmentation to arbitrary type. Interestingly, their conditions for controllability, and therefore for pole placement, are identical to those above. Earlier work along these lines had been conducted independently by Smith and Davison [9].

An alternate viewpoint on the problem of system type is to consider the system to be asymptotically decoupled for a certain class of inputs. In this case, the problem is similar to those posed and solved by Davison [10] and Huang [11].

A distinctively different approach to the problem of determining conditions for zero steady-state tracking error is given by the geometric theory of linear multivariable control [12]. Conditions, expressed in terms of invariant subspaces, which guarantee the desired tracking properties have been obtained by Bhattacharyya and Pearson [13,14], Wonham [15], Wonham and Pearson [16], and Francis and Wonham [17]. System type is not dealt with explicitly in these studies, but it would seem that an intimate connection should exist.

The approach taken here is based on a generalization of the Hosoe-Ito definition. It leads to new characterizations of system type and provides methods for determining type from either the open-loop transfer function matrix G(s) or the state-space representation (A,B,C,D) which are simpler to apply than those in previous works. These topics are treated in Sections II and III. In Section IV, new results are obtained for the characterization and identification of type for composite systems. The conclusion is given in Section V. In a companion paper [18], the theory and methods for achieving a specified type via either pre- or post-compensation are presented.

II. TYPE $[\ell_1 \ell_2 \ldots \ell_m]$ TRANSFER FUNCTION MATRICES

The following is a generalization of the Hosoe-Ito [6] definition of system type.

<u>Definition 1</u>

A unity feedback multivariable linear system is called type $[\ell_1 \ell_2 \ldots \ell_m]$ if each ℓ_i, $i=1,2,\ldots,m$, is the largest integer such that the $m\times m$ open-loop transfer function matrix $G(s)$ can be factored into

$$G(s) = H(s)\, G'(s)$$

where

$$H(s) = \mathrm{diag}\left[\frac{1}{s^{\ell_1}}\ \frac{1}{s^{\ell_2}}\ \cdots\ \frac{1}{s^{\ell_m}}\right]$$

and $G'(s)$ has no zeros at the origin. Here, $G(s)$ is termed a type $[\ell_1 \ell_2 \ldots \ell_m]$ transfer function matrix. □

This definition generalizes that in [6] to the extent of allowing $\ell_i<0$. In terms of the steady-state tracking properties, it is of no consequence whether the ℓ_i are less than zero or equal to zero. The magnitude of ℓ_i, when ℓ_i is less than zero, however, will be of importance in the sequel when the problem of system type for series connections of systems is considered.

It is worthwhile to note at this point that system type for multivariable systems is defined in terms of a left divisor. It will be shown that this leads to meaningful results for steady-state errors. The following simple example shows that a *right divisor* will give different results, in fact, results which are erroneous.

<u>Example 1</u>. Let

$$G(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

Factoring $G(s)$ according to the definition yields

$$G(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

Thus $G(s)$ is the open-loop transfer function matrix of a type $[1\ \ 0]$ system.

If now an attempt is made to find an $H(s)$ which is a right divisor, none can be found except the identity. This would indicate that $G(s)$ is type $[0\ \ 0]$. □

With system type as specified by Definition 1, it will be shown below that a stable servomechanism with a type $[\ell_1 \ell_2 \ldots \ell_m]$ open-loop transfer function matrix yields zero steady-state error for input signals of the form

$$y_d(t) = \begin{bmatrix} \dfrac{a_1 t^{k_1}}{k_2!} \\ \dfrac{a_2 t^{k_2}}{k_2!} \\ \vdots \\ \dfrac{a_m t^{k_m}}{k_m!} \end{bmatrix} \qquad (3)$$

where for $\ell_i>0$, the a_i are arbitrary real constants and $0\le k_i<\ell_i$ with the k_i integers; and for $\ell_i\le 0$, $a_i=k_i=0$.

The following definition and lemma will be useful in the proof of the main theorem of this section.

<u>Definition 2.</u>

With $G(s)$, $H(s)$, and $G'(s)$ as in Definition 1, $\tilde{H}(s)$ and $\hat{H}(s)$ are defined by

$$\tilde{H}(s) = \mathrm{diag}\left[\frac{1}{s^{\ell_1^-}}\ \frac{1}{s^{\ell_2^-}}\ \cdots\ \frac{1}{s^{\ell_m^-}}\right]$$

$$\hat{H}(s) = \mathrm{diag}\left[\frac{1}{s^{\ell_1^+}}\ \frac{1}{s^{\ell_2^+}}\ \cdots\ \frac{1}{s^{\ell_m^+}}\right]$$

where

$$\ell_i^+ = \begin{cases} \ell_i & \text{if } \ell_i > 0 \\ 0 & \text{if } \ell_i \le 0 \end{cases} \quad \text{and} \quad \ell_i^- = \begin{cases} \ell_i & \text{if } \ell_i < 0 \\ 0 & \text{if } \ell_i \ge 0 \end{cases}$$

so that

$$H(s) = \hat{H}(s)\, \tilde{H}(s) = \tilde{H}(s)\, \hat{H}(s) \qquad \square$$

Note that $\tilde{H}(s)$ has only nonnegative powers of s as elements, while $\hat{H}(s)$ involves only nonpositive powers of s. Now $G(s)$ can be factored as

$$G(s) = H(s)\, G'(s) = \hat{H}(s)\, \tilde{G}(s)$$

where $\tilde{G}(s) = \tilde{H}(s)\, G'(s)$.

<u>Lemma 1</u>. With $G(s) = \hat{H}(s)\, \tilde{G}(s)$ as above, $\Delta_G = \Delta_{\hat{H}}\, \Delta_{\tilde{G}}$.

<u>Proof</u>. From Lemma 1 of [4] it follows that Δ_G divides $\Delta_{\hat{H}}\Delta_{\tilde{G}}$. Thus it suffices to show that $\deg(\Delta_G) = \deg(\Delta_{\hat{H}}) + \deg(\Delta_{\tilde{G}})$. By the definition of $H(s)$ and $\hat{H}(s)$, each factor of s in $\Delta_{\hat{H}}$ must correspond to factoring a pole of $G(s)$ into $H(s)$. Otherwise, Δ_G would remain constant while the determinant would lose a factor s^{-1}, adding a zero at $s=0$, which is not allowed. Thus, each factor s in $\Delta_{\hat{H}}$ reduces the order of Δ_G by one when going from Δ_G to form $\Delta_{\tilde{G}}$. Also, the factorization cannot introduce any other new poles. Hence,

$$\deg(\Delta_G) = \deg(\Delta_{\hat{H}}) + \deg(\Delta_{\tilde{G}}) \qquad \square$$

A simple modification of the proof of this lemma shows that $\Delta_G = \Delta_H \Delta_{\tilde{G}}$.

Now the theorem on steady-state error can be stated and proved.

Theorem 1.

Let G(s) by a type $[\ell_1 \ell_2 \ldots \ell_m]$ transfer function matrix. Then the stable unity feedback system having G(s) as its open-loop transfer function matrix will track inputs of the form given in Eq. (3) with zero steady-state error.

Proof. The Laplace transform of Eq. (3) yields

$$Y_d(s) = \begin{bmatrix} \frac{a_1}{s^{k_1+1}} & \frac{a_2}{s^{k_2+1}} & \cdots & \frac{a_m}{s^{k_m+1}} \end{bmatrix}^T$$

where $(\ )^T$ denotes the transpose. Then the Laplace transform of the system error is

$$E(s) = [I + G(s)]^{-1} Y_d(s) \tag{4}$$

Now with H(s) as given in Definition 2, it is noted that

$$\lim_{s\to 0} \hat{H}^{-1}(s)\, Y_d(s) = z_o \tag{5}$$

where z_o is a finite m-vector.

Thus if $[I + G(s)]^{-1}$ is factored as

$$[I + G(s)]^{-1} = \{\hat{H}(s)\cdot[\hat{H}^{-1}(s) + \tilde{G}(s)]\}^{-1}$$
$$= [\hat{H}^{-1}(s) + \tilde{G}(s)]^{-1}\hat{H}^{-1}(s)$$

and it can be shown that

$$\lim_{s\to 0}[\hat{H}^{-1}(s) + \tilde{G}(s)]^{-1} = G_o \tag{6}$$

where G_o has finite elements, then the combination of Eqs. (4), (5), and (6) can be used to show that

$$\lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s \cdot \lim_{s\to 0} E(s)$$
$$= \lim_{s\to 0} s \cdot \lim_{s\to 0} [\hat{H}^{-1}(s) + \tilde{G}(s)]^{-1}$$
$$\bullet \lim_{s\to 0} [\hat{H}^{-1}(s)\, Y_d(s)]$$
$$= \lim_{s\to 0} s\, G_o\, z_o = 0 \tag{7}$$

Finally, Eq. (7) can be used in the final-value theorem of the Laplace transform to prove the present theorem.

First, Eq. (6) must be proved. By the assumed stability of the unity feedback system, the zeros of the polynomial Δ_G det $[I + G(s)]$ must be in the open left-half complex plane [20]. Then

$$\lim_{s\to 0} \Delta_G \det [I + G(s)] \neq 0 \tag{8}$$

Using Lemma 1,

$$\Delta_G \det [I + G(s)] = \hat{\Delta}_H \Delta_{\tilde{G}} \det[I + G(s)]$$
$$= \hat{\Delta}_H \Delta_{\tilde{G}} \det [H(s)]$$
$$\bullet \det[\hat{H}^{-1}(s) + \tilde{G}(s)] \tag{9}$$

By construction of $\hat{H}(s)$,

$$\hat{\Delta}_H \det [\hat{H}(s)] = 1 \tag{10}$$

Combining Eqs. (10), (9) and (8) yields

$$\lim_{s\to 0} \Delta_{\tilde{G}} \det [\hat{H}^{-1}(s) + \tilde{G}(s)] \neq 0 \tag{11}$$

With the product in Eq. (11) nonzero, then Eq. (6) will be verified if the product of $\Delta_{\tilde{G}}$ with each cofactor of $[\hat{H}^{-1}(s) + \tilde{G}(s)]$ is a polynomial in s. Since $\hat{H}^{-1}(s)$ has no poles, the poles of $[\hat{H}^{-1}(s) + \tilde{G}(s)]$ are a subset of the poles of $\tilde{G}(s)$. It then follows, since $\Delta_{\tilde{G}}$ contains a factor for each pole of $\tilde{G}(s)$, that $\Delta_{\tilde{G}}$ times any cofactor of $[\hat{H}^{-1}(s) + \tilde{G}(s)]$ is a polynomial. As a result,

$$\lim_{s\to 0}[\hat{H}^{-1}(s) + \tilde{G}(s)]^{-1}_{ij}$$
$$= \lim_{s\to 0} \frac{\Delta_{\tilde{G}} \text{ cofactor } [\hat{H}^{-1}(s) + \tilde{G}(s)]_{ij}}{\Delta_{\tilde{G}} \det [\hat{H}^{-1}(s) + \tilde{G}(s)]}$$
$$= \text{a finite number for all } i, j$$

where $1 \le i \le m$ and $1 \le j \le m$. Thus Eq. (6) is verified and Eq. (7) holds. Using the final-value theorem, the steady-state error is zero if the limit exists. However, by the assumed stability of the unity feedback system, the limit does exist and the theorem is proved. □

As an immediate consequence of Lemma 1, one has the following proposition concerning bounds on the type of a transfer function type.

Proposition 1. Let G(s) be a type $[\ell_1 \ell_2 \ldots, \ell_m]$ transfer function matrix. Also, let

$$L^+ = \sum_{i=1}^{m} \ell_i^+ \text{ and } L^- = \sum_{i=1}^{m} \ell_i^-. \text{ Then,}$$

(i) L^+ is less than or equal to the multiplicity of the pole of G(s) at the origin

and

(ii) L^- is greater than or equal to the multiciplicity of the zero of G(s) at the origin.

Proof. (i) If L^+ is greater than the multiplicity of the pole of G(s) at the origin, then zeros at the origin have been added to G'(s) contradicting the definition.

(ii) If L^- is less than the multiplicity of the zero of G(s) at the origin, then a zero would remain at the origin in G'(s) again contradicting the definition. □

III. IDENTIFICATION OF TRANSFER FUNCTION TYPE

If the dimension m of the transfer function matrix G(s) is small, say two or three, the type of the system can be found via trial and error using Definition 1. However, for larger values of m, the procedure would be extremely tedious since at each stage the characteristic polynomial must be computed. In this section, two simpler methods of computing the system type are given. The first method gives the system type from a simple observation of one of the factors in a coprime factorization of the transfer function matrix. The other method is an algorithm which gives the system type based on rank conditions using the state-space representation (A,B,C,D) which is assumed to be a minimal realization. These methods are simpler than the rank conditions in [6], especially if there are large power of s in the denominator of any element of G(s).

Both methods considered give the system type only to the point of determining the ℓ_i^+. If a particular element ℓ_i is actually negative, these methods will indicate that the ℓ_i is equal to zero. Thus, only the elements of H(s) are determined. This is, however, the only part which is of concern when the steady-state properties are considered as was demonstrated in the proof of Theorem 1.

The first method is presented by way of the following theorem.

Theorem 2.

Let G(s) be the open-loop transfer function matrix of a unity feedback system and let $D^{-1}(s)N(s)$ be a coprime factorization of G(s). Then the unity feedback system is type $[\ell_1^+ \ell_2^+ \ldots \ell_m^+]$ where the ℓ_i^+, i = 1,2, ..., m, are the greatest power of s such that each element of column i of D(s) is divisible by s.

Proof. From Definition 1, each ℓ_i^+ is maximal such that no zeros are introduced in $\tilde{G}(s)$ by the factorization $G(s) = \hat{H}(s)\tilde{G}(s)$. Further, $\tilde{G}(s)$ can be expressed in the form

$$\tilde{G}(s) = \hat{H}^{-1}(s)\, G(s)$$
$$= \hat{H}^{-1}(s)\, D^{-1}(s)\, N(s)$$

where $D^{-1}(s)N(s)$ is any coprime factorization of G(s), and $\hat{H}^{-1}(s)$, N(s) and D(s) are all m×m polynomial matrices. The polynomial system matrix for G(s) is

$$P(s) = \begin{bmatrix} D(s) & N(s) \\ \hat{H}^{-1}(s) & 0 \end{bmatrix}$$

The zeros of $\tilde{G}(s)$ are the zeros of det P(s) after the decoupling zeros [19] have been removed. Since D(s) and N(s) are coprime, the only decoupling zeros are output decoupling zeros. Let

$$P_1(s) = \begin{bmatrix} D_1(s) & N(s) \\ \hat{H}_1^{-1}(s) & 0 \end{bmatrix}$$

denote P(s) with its output decoupling zeros removed.

The zeros of $\tilde{G}(s)$ are the solutions of

$$0 = \det P_1(s)$$
$$= \det \hat{H}_1^{-1}(s) \cdot \det N(s)$$

So in order that no zeros be introduced into $\tilde{G}(s)$, all of the factors of s in $\hat{H}^{-1}(s)$ must be output decoupling zeros and hence removable. By repeated application of the lemma in the Appendix to $[D^T(s) \mid \hat{H}^{-1}(s)^T]$ with $\alpha=0$, the maximal powers of s which can be removed from P(s) are the largest powers of s, say ℓ_i^+, which divide each element of column i of D(s). So the largest power of s in entry i of $\hat{H}^{-1}(s)$ such that no zeros are introduced in $\tilde{G}(s)$ and the largest power of s which divides each element of column i of D(s) are the same, and the theorem is proved. □

Algorithms for determining a coprime factorization of a rational transfer function matrix can be found in [22]. The following example demonstrates the method of determining system type which is described in the above theorem.

Example 2. Let

$$G(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s} & \frac{1}{s+2} \\ 0 & \frac{1}{s+2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{s^2(s+1)} & 0 & \frac{1}{s} \end{bmatrix}$$

A coprime factorization yields

$$G(s) = \begin{bmatrix} \frac{1}{s(s+2)} & 0 & 0 \\ 0 & \frac{1}{(s+1)(s+2)} & 0 \\ 0 & 0 & \frac{1}{s^2(s+1)} \end{bmatrix}$$

$$\times \begin{bmatrix} s+2 & s+2 & s \\ 0 & s+1 & 1 \\ 1 & 0 & s(s+1) \end{bmatrix}$$

Then D(s) is

$$D(s) = \begin{bmatrix} s(s+2) & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & s^2(s+1) \end{bmatrix}$$

and it follows that G(s) is type [1 0 2]. □

The second method of determining transfer function matrix type is for the state-space representation (A,B,C,D) which is assumed to be a minimal realization with the four matrices n×m, n×r, m×n, and m×r, respectively. Rather than a direct characterization of type, the following theorem presents an algorithm which determines the value of ℓ_i^+ for each i.

Theorem 3.

If the state-space quadruple (A,B,C,D) represents the open-loop elements of a stable m-input/m-output unity feedback system, then the following algorithm will determine the system type.

Algorithm. Let $Q = [C^T \vdots A^T C^T \vdots A^{2T} C^T \vdots \ldots \vdots A^{n-1^T} C^T]^T$, i.e., the observability matrix, and let

$$Q_i^j = \left[C_i^{j^T} \vdots A^T C_j^{i^T} \vdots A^{2^T} C_i^{j^T} \vdots \ldots \vdots A^{n-1^T} C_i^{j^T} \right]^T$$

where C_i^j is the matrix C with row i replaced by row i of CA^j. With these matrices, perform the following steps:

(1) Let $p \triangleq$ rank Q
(2) Set row counter $i = 0$
(3) Increment row counter i by 1
(4) Set $j = 0$
(5) Increment j by 1
(6) Form Q_j^i
(7) Set $k =$ rank Q_j^i
(8) If $k + j = p$, go to (5)
(9) Set $\ell_i = j - 1$
(10) If $i < m$, go to (3)
(11) End.

Proof. In the proof of Theorem 2, it was shown that the ℓ_i of the system type vector are the largest powers of s such that multiplication of row i of G(s) by s introduced exactly ℓ_i output decoupling zeros. Multiplication by larger powers would not introduce any more decoupling zeros. The effect of multiplication of row i of G(s) by s to some power when G(s) is given by

$$G(s) = C(sI - A)^{-1}B + D$$

is developed as follows.

The result of multiplying row i of G(s) by s^j is given by row i of

$$s^j\, C(sI - A)^{-1}\, B + s^j D \tag{12}$$

By using the identity $(sI - A)(sI - A)^{-1} = I$, the result

$$s(sI - A)^{-1} = A(sI - A)^{-1} + I \tag{13}$$

is obtained. Repeated application of Eq. (13) to Eq. (12) yields the result

$$s^j\, C(sI - A)^{-1}\, B + s^j D = CA^j (sI - A)^{-1} B + \sum_{k=0}^{j-1} CA^k\, B s^{j-k-1} + s^j D$$

The system matrix for G(s) with row i multiplied by s^j is then

$$P_i^j(s) = \begin{bmatrix} sI-A & B \\ C_i^j & D_i^j(s) \end{bmatrix}$$

where C_i^j is given by the algorithm and $D_i^j(s)$ is D with row i replaced by row i of

$$s^j\, D + \sum_{k=0}^{j-1} s^{j-k-1}\, CA^k B$$

From Theorem 8.1, Chapter 2 of Rosenbrock [19], the rank defect of the observability matrix Q is the number of output decoupling zeros. Thus, when j is incremented by 1 in the algorithm, this corresponds to multiplication of row i of G(s) by one higher power of s. Then if the rank Q_i^j is decreased by 1 when j is increased by 1, an output decoupling zero was added. The algorithm thus finds the largest j such that the rank of Q_i^j decreases at each step which, as was seen in the proof of Theorem 2, is the ℓ_i of the matrix transfer function.

As was also noted in the proof of Theorem 2, the number of output decoupling zeros introduced by element i of $H^{-1}(s)$ is independent of the others. This justifies the procedure in the algorithm of looking at one row at a time. □

The following example demonstrates the algorithm.

Example 3. Let the open-loop elements of a unity feedback system be defined by the quadruple

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using the algorithm, one finds that

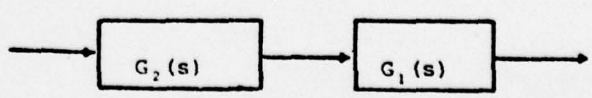
$$\text{rank } Q = 3$$
$$\text{rank } Q_1^1 = 2$$
$$\text{rank } Q_1^2 = 1$$
$$\text{rank } Q_1^3 = 1 \text{ (hence } \ell_1 = 2)$$
$$\text{rank } Q_2^1 = 2$$
$$\text{rank } Q_2^2 = 2 \text{ (hence } \ell_2 = 1)$$

Thus, [A,B,C,D] represents a type [2 1] transfer function matrix. This can easily be verified in this case since

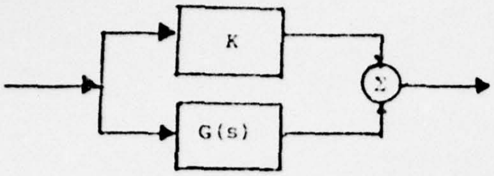
$$G(s) = \begin{bmatrix} \frac{s+1}{s^2} & 0 \\ \frac{1}{s} & \frac{1}{s} \end{bmatrix} \quad \square$$

IV. SYSTEM TYPE FOR COMPOSITE SYSTEMS

In this section, some results are obtained for system type of composite systems. The first result is a bound on the system type for the series connection [see Fig. 2(a)] in terms of the transfer function type of the individual transfer function matrices. The second result gives conditions under which a transfer function type is maintained when it is connected in parallel with a constant gain matrix [see Fig. 2(b)].



(a) Series connection



(b) Parallel connection

Fig. 2. Composite system connections.

In the first instance, one has the following result.

<u>Theorem 4.</u>

If $G_1(s)$ is type $[\ell_1 \ell_2 \ldots \ell_m]$ and $G_2(s)$ is type $[\ell_1' \ell_2' \ldots \ell_m']$ then the series combination $G_1(s)G_2(s)$ is at least type $[\ell_1+\ell\ \ \ell_2+\ldots \ell_m+\ell]$ where $\ell = \min [\ell_1' \ell_2' \ldots \ell_m']$. (By at least type $[\ell_1+\ell\ \ \ell_2+\ell \ldots \ell_m+\ell]$, it is meant that each $\ell_i'' \geq \ell_i+\ell$ where ℓ_i'' is the type for the i^{th} input-output pair of the composite system.) The proof follows from Def. 1. □

The following corollary is an immediate consequence of the theorem.

<u>Corollary 1</u>. If G(s) is type $[\ell_1 \ell_2 \ldots \ell_m]$ and K(s) is a dynamic compensator which is used to stabilize the unity feedback system with G(s)·K(s) as its open-loop transfer function matrix, then the composite system is at least type $[\ell_1 \ell_2 \ldots \ell_m]$ provided that K(s) has no zeros at the origin.

<u>Remark</u>. Corollary 1 was stated in [6] without the condition that K(s) have no zeros at the origin. A simple example in the single input/single-output case shows the necessity for this requirement.

The second result of this section pertains to a type $[\ell_1 \ell_2 \ldots \ell_m]$ transfer function matrix which has a constant gain matrix in parallel [see Fig. 2(b)]. This result is used in [23] in a study of stochastic multivariable servomechanisms.

<u>Theorem 5.</u>

Let $G(s) = D^{-1}(s)N(s)$ be a type $[\ell_1 \ell_2 \ldots \ell_m]$ transfer function matrix and K an m×m constant gain matrix. Then G(s)+K is at least type $[\ell_1 \ell_2 \ldots \ell_m]$ if and only if det [N(s)+D(s)K] has no zeros at the origin.

<u>Proof.</u> Since K has no poles, the poles of the parallel combination G(s)+K are simply the poles of G(s). Now G(s)+K can be factored as

$$G(s) + K = H(s)[G'(s) + H^{-1}(s)K] \qquad (16)$$

and Eq. (16) can be written as

$$\begin{aligned} G(s) + K &= H(s)[G'(s) + H^{-1}(s)K] \\ &= H(s)\tilde{D}^{-1}(s)[N(s)+\tilde{D}(s)H^{-1}(s)K] \\ &= H(s)\tilde{D}^{-1}(s)[N(s)+D(s)K] \end{aligned}$$

Hence, G(s)+K is at least type $[\ell_1 \ell_2 \ldots \ell_m]$ if and only if det [N(s)+D(s)K] has no zeros at s = 0. □

<u>Corollary 2</u>. If G(s) is type $[\ell_1 \ell_2 \ldots \ell_m]$ and $\ell_i > 0$ for every i, then G(s)+K is at least type $[\ell_1 \ell_2 \ldots \ell_m]$.

<u>Proof</u>. Using the property of D(s) given in Theorem 2, and the assumed values for the ℓ_i, every element of the product D(s)K is divisible by s. Then the constant term in the equation det [N(s)+D(s)K]=0 must be the same as the constant term in the equation det N(s)=0. Since N(s) has no zeros at the origin, the constant term is nonzero. Therefore det [N(s)+D(s)K] has no zeros at the origin. □

V. CONCLUSION

In this paper, a theory of system type for linear multivariable servomechanisms, which is based on a generalization of the Hosoe-Ito [6] definition of system type, has been presented. The more general definition permits the development of results for the determination of system type for composite systems.

New characterizations of system type have been formulated and two systematic methods for determining system type have been derived. The first method is directly suited for application to the transfer function matrix (frequency domain) description of the system, the second for the state-space (time-domain) representation. Both are easier to apply than those in previous studies and are algorithmic in nature. The second method is the only known result which treats multivariable system type directly in terms of the state-space quadruple (A,B,C,D) without need to compute first the corresponding transfer function matrix.

The results here on system type for composite multivariable linear systems in terms of the type for the individual subsystems are also new, and have found applications elsewhere [18,23].

APPENDIX

It is known [19] that the system matrix

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}$$

gives rise to the transfer function matrix

$$G(s) = V(s)\,T^{-1}(s)U(s) + W(s)$$

and conversely to within strict system equivalence where T,U,V, and W are m×n, n×r, m×n, and m×r, respectively.

A system matrix which contains decoupling zeros can be reduced to one of least order by removal of the decoupling zeros. The following lemma concerning reduction of order is needed in the proof of Theorem 2.

<u>Lemma</u>. Let $U(s) = \mathrm{diag}\,[1\ \ 1 \ldots 1\ (s-\alpha)\ 1 \ldots 1]$ where $(s-\alpha)$ is the i^{th} diagonal element and α is any complex number. Then rank $[T(\alpha)\ \ U(\alpha)] < m$ if and only if every element

of i of T(s) is divisible by $s - \alpha$.

Proof. Sufficiency is obvious. Necessity comes from the fact that if any element of row i is not divisible by $s - \alpha$, then at $s = \alpha$ the remaining $m - 1$ columns of $U(\alpha)$ and one column of $T(\alpha)$ which has a nonzero element in row i form a linearly independent set. Thus, rank $[T(s)\ U(s)] = m$ for all complex s if not all elements of row i of T(s) are divisible by $s - \alpha$. □

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A theory of system type for linear multivariable servomechanisms is developed. New characterizations of system type are formulated and two systematic methods for determining system types are derived. Both methods are algorithmic and easier to apply than the techniques given in earlier studies. The first applies to the case where the open-loop system is described by its transfer function matrix, the second where its time domain state-space representation is given. New results are also presented for characterizing and identifying the system type for certain composite systems from a knowledge

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